<u>Arithmetic and Geometric sequences and series</u>

1. Find the range of values of x for which the following geometric series converges and find the sum to infinity for such values of x,

$$1-\frac{x}{1-x}+\left(\frac{x}{1-x}\right)^2-\left(\frac{x}{1-x}\right)^3+\ldots$$

2. Determine the common range of values of x for which the two infinite geometric series

$1-x+x^2-x^3+\ldots$, $\quad 1+\frac{x}{1+x}+\left(\frac{x}{1+x}\right)^2+\left(\frac{x}{1+x}\right)^3+\ldots$ have sums to infinity.

If for any value of x in this range, the sum to infinity are S_1 and S_2, prove that $S_1 S_2 = 1$.

3. The p^{th} term of a sequence is P, the q^{th} terms is Q, and the r^{th} is R.

Show that if the sequence is arithmetical, then $P(q-r)+Q(r-p)+R(p-q)=0$,
and that if it is geometrical, then $(q-r)\log P+(r-p)\log Q+(p-q)\log R=0$.

4. The sum of the first n terms of a geometrical sequence is 255 and the sum of their reciprocal is $255/128$. If the first term is 1, find n and the common ratio.

5. "The sum to infinity of a convergent geometric series, whose first term is a and whose common ratio is r, is $a/(1-r)$." Explain as clearly as you can what is meant by "convergent series" and "sum to infinity of a convergent series"; and prove the truth of the above statement.
Are the following series convergent or not? Give your reasons fully:

(a) $2+\frac{4}{3}+\frac{8}{9}+\frac{16}{27}+\ldots$ **(b)** $1-1+1-1+1-\ldots$

6. The sum of the first n terms of an arithmetical series $u_1+u_2+u_3+\ldots$ is denoted by S_n.
If $u_m = 4$, $u_{4m} = 24$, and $S_{4m} = 44\,S_m$, find the values of u_1 and m.

7. If the sum of the first n terms of an arithmetical sequence is $2n$, and the sum of the first $2n$ terms is n, find the sum of the first $4n$ terms.

8. Find the sum of all the positive integers less than 1000 which are not multiples of 3.

9. If the positive integers are added in groups as shown, $1+(2+3)+(4+5+6)+(7+8+9+10)+\ldots$, find the first integer of the n^{th} group, unity being counted as the first group. Find also the sum of the integers forming the n^{th} group.

10. If m is a positive integer, show that the sum of the arithmetical series

$(2m+1)+(2m+3)+(2m+5)+\ldots+(4m-1)$ is divisible by 3.

If m is also even, show that this sum is divisible by 12.

11. The sum of the squares of three positive numbers in arithmetical sequence equals 165 . The sum of the numbers is 21 . Find the numbers.

12. Let a^2 , b^2 , c^2 form an arithmetic sequence. Prove that the quantities also form an arithmetic sequence.

13. Prove that if a , b and c are respectively the p^{th}, q^{th} and r^{th} terms of an arithmetic sequence, then $(q – r) a + (r – p) b + (p – q) c = 0$.

14. Let in an arithmetic sequence $a_p = q , a_q = p$ (a_n is the n^{th} terms of the sequence.) Find a_m .

In an arithmetic series, $S_p = q$, $S_q = p$ (S_n is the sum of the first n terms of the series.). Find S_{p+q} .

15. Let in an arithmetic series $S_p = S_q$. Prove that $S_{p+q} = 0$.

16. Let in an arithmetic series $\dfrac{S_m}{S_n} = \dfrac{m^2}{n^2}$. Prove that $\dfrac{a_m}{a_n} = \dfrac{2m-1}{2n-1}$.

17. Let $a_1 , a_2 , \dots , a_n$ form an arithmetic sequence and $a_1 = 0$. Simplify the expression :

$$S = \frac{a_3}{a_2} + \frac{a_4}{a_3} + \dots + \frac{a_n}{a_{n-1}} - a_2\left(\frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_{n-2}}\right) .$$

18. Prove that in any arithmetic sequence $a_1 , a_2 , \dots , a_n$, we have :

$$S = \frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \dots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}$$

19. Show that in any arithmetic sequence $a_1 , a_2 , \dots , a_n$, we have :

$$S = a_1^{\,2} - a_2^{\,2} + a_3^{\,2} - a_4^{\,2} + \dots + a_{2k-1}^{\,2} - a_{2k}^{\,2} = \frac{k}{2k-1}\left(a_1^{\,2} - a_{2k}^{\,2}\right) .$$

20. Let $S(n)$ be the sum of the first n terms of an arithmetic series. Prove that

(i) $S(n +3) – 3S(n + 2) + 3S(n + 1) – S(n) = 0$

(ii) $S(3n) = 3[S(2n) – S(n)]$.

21. Let S_n be the sum of the first n terms of a geometric series.

Prove that $S_n (S_{3n} – S_{2n}) = (S_{2n} – S_n)^2$.

22. Let $a_1, a_2, \dots, a_n$ are real, then the equality:

$$(a_1^2 + a_2^2 + \dots + a_{n-1}^2)(a_2^2 + a_3^2 + \dots + a_n^2) = (a_1 a_2 + a_2 a_3 + \dots + a_{n-1} a_n)^2$$

is possible if and only if $a_1, a_2, \dots, a_n$ form a geometric sequence. Prove this.

23. Let $a_1, a_2, \dots, a_n$ be a geometric sequence with ratio r and let $S_m = a_1 + a_2 + \dots + a_m$.
Find simpler expression for the following sums:

(i) $S_1 + S_2 + \dots + S_n$;

(ii) $\dfrac{1}{a_1^2 - a_2^2} + \dfrac{1}{a_2^2 - a_3^2} + \dots + \dfrac{1}{a_{n-1}^2 - a_n^2}$;

(iii) $\dfrac{1}{a_1^k + a_2^k} + \dfrac{1}{a_2^k + a_3^k} + \dots + \dfrac{1}{a_{n-1}^k + a_n^k}$.

24. Prove that in any arithmetic sequence, whose common difference is not equal to zero, the product of two terms equidistant from the extreme terms is the greater the closer these terms are to the middle.

25. An arithmetic and a geometric sequence with positive terms have the same number of terms and equal extreme terms. For which of them is the sum of terms greater.

26. The first two terms of an arithmetic and a sequence with positive terms are equal. Prove that all other terms of the arithmetic sequence are not greater than the corresponding terms of the geometric sequence.

27. $a_1, a_2, \dots, a_n$ are in geometric sequence whose common ratio is r. Express in terms of a_1, r and n, the products :

$a_1 a_3 \dots a_{2n-1}$

$a_2 a_4 \dots a_{2n}$.